XVII. *On the Method of correspondent Values*, &c. *By* Edward Waring, *M. D. F. R. S. and* Lucasian *Professor of the Mathematics at* Cambridge.

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I.

1. IN the year 1762 I published a method of finding when two roots of a given equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c.$=0$ are equal, by finding the common divisors of the two quantities $a^n - pa^{n-1} + qa^{n-2} -$ &c., and $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} -$ &c., and observed if they admitted only one simple divisor $(a - A)$, then two roots were only equal; if a quadratic $(a^2 - Aa + B)$, then two roots of the equation became twice equal; if a cubic $(a^3 - Aa^2 + Ba - C)$, then two roots became thrice equal; and so on: or, to express in more general terms what follows from the same principles, if the common divisor be $\overline{a-b}^r \times \overline{a-c}^s \times \overline{a-d}^t \times$ &c., then $r+1$ roots of the given equation will be b, $s+1$ roots will be c, $t+1$ will be d, &c.; and it immediately follows, from the principles delivered in the second edition of the same Book, published in 1770, that to find when $r+1$, $v+1$, $t+1$, &c. roots are respectively equal requires $r+s+t$, &c. equations of condition, which are deducible from the well known method of finding the common divisors of two quantities in this case of $a^n - pa^{n-1} + qa^{n-2} -$ &c., $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} -$ &c. of the terms of their remainders, &c.

In

In the book above mentioned the equations of condition are given, which difcover when two roots are equal in the equations $x^3 - px^2 + qx - r = 0$, $x^4 + qx^2 - rx + s = 0$, $x^5 + qx^3 - rx^2 + sx - t = 0$, in the two latter equations the fecond term is wanting, which may eafily be exterminated; but it may as eafily be reftored by fubftituting for q, r, s, &c. in the equation of condition found the quantities refulting from the common transformation of equations to deftroy the fecond term.

2. Another rule contained in the fame Book is the fubftitution of the roots of the equation $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c. = 0$ refpectively for a in the quantity $a^n - pa^{n-1} + qa^{n-2} - \&c.$, and multiplication of all the quantities refulting into each other; their content will give the equation of condition, when two roots are equal.

Mr. HUDDE firft difcovered, that if the fucceffive terms of the given equation are multiplied into an arithmetical feries, the refulting equation will contain one of any two equal roots, and m of the $\overline{m+1}$ equal roots in the given equation.

3. If 3, 4, 5, .. r roots of the equation are equal, find a common divifor of 3, 4, 5, .. r of the fubfequent quantities $a^n - pa^{n-1} + qa^{n-2} - \&c.$, $na^{n-1} - \overline{n-1}pa^{n-2} + \overline{n-2}qa^{n-3} - \&c.$, $n \,.\, \overline{n-1}a^{n-2} - \overline{n-1} \,.\, \overline{n-2}pa^{n-3} + \overline{n-2} \,.\, \overline{n-3}qa^{n-4} - \overline{n-3} \,.\, \overline{n-4}ra^{n-5} + \&c.$, $n \,.\, \overline{n-1} \,.\, \overline{n-2}a^{n-3} - \overline{n-1} \,.\, \overline{n-2} \,.\, \overline{n-3}pa^{n-4} + \overline{n-2} \,.\, \overline{n-3} \,.\, \overline{n-4}qa^{n-5} - \&c.$, ... $n \,.\, \overline{n-1} \,.\, \overline{n-2} \,..\, \overline{n-r+2}a^{n-r+1} - \overline{n-1} \,.\, \overline{n-2} \,..\, \overline{n-r+1}pa^{n-r} + \&c.$; which will probably be beft done by dividing all the preceding quantities by the quantity of the leaft dimenfion of a, and the divifor and all the remainders by that quantity which has the leaft dimenfions amongft them; and fo on: there will refult 2, 3, 4, .. $r-1$ equations of condition; and in this cafe it is

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observed, in the before-mentioned Book, that (if the common divisor be $(a - A)$) it will once only admit of 3, 4, 5, . . r equal roots; if it be a quadratic, then it will twice admit of those equal roots; and so on.

4. If the roots of the equation of the least dimensions be substituted for a in the remaining equations, and each of the resulting values of the same equation be multiplied into each other, there will result the $r - 1$ equations of condition: and the same may be deduced also from the several equations conjointly.

The equations of conditions found by the first method, if the divisions were not properly instituted, may admit of more rational divisors than necessary, of which some are the equations of conditions required.

2.

1. In the year 1776, I published in the Meditationes Analyticæ a new method of differences for the resolution of the following problem.

Given the sums of a swiftly converging series $ax + bx^2 + cx^3 + dx^4 + \&c.$, when the values of x are respectively π, ρ, ς, &c.; to find the sum of the series when x is τ, that is, given $S\pi = a\pi + b\pi^2 + c\pi^3 + d\pi^4 + \&c$, $S\varrho = a\varrho + b\rho^2 + c\rho^3 + \&c.$, $S\sigma = a\sigma + b\sigma^2 + c\sigma^3 + \&c.$ &c.; to find $S\tau = a\tau + b\tau^2 + c\tau^3 + \&c.$

To resolve this problem I multiplied the quantities, $S\pi$, $S\rho$, $S\sigma$, &c. respectively into unknown co-efficients α, β, γ, &c. and there resulted

απα

$\alpha\pi a + \alpha\pi^2 b + \alpha\pi^3 c + \&c.$
$\beta\rho a + \beta\rho^2 b + \beta\rho^3 c + \&c.$
$\gamma\sigma a + \gamma\sigma^2 b + \gamma\sigma^3 c + \&c.$
&c. &c. &c.

and then made the ſum of each of the terms reſpectively equal to its correſpondent term of the quantity $\tau a + \tau^2 b + \tau^3 c + \&c.$, and conſequently $\alpha\pi + \beta\rho + \gamma\sigma + \&c. = \tau$, $\alpha\pi^2 + \beta\rho^2 + \gamma\sigma^2 + \&c. = \tau^2$, $\alpha\pi^3 + \beta\varrho^3 + \gamma\sigma^3 + \&c. = \tau^3$, &c. I aſſumed as many equations of this kind as there were given values π, ϱ, σ, &c. of x; and conſequently as many equations reſulted as unknown quantities α, β, γ, &c.; whence, by the common reſolution of ſimple equations, or more eaſily from differences, can be found the unknown quantities α, β, γ, &c., and thence the equation ſought $\alpha \times S\pi + \beta \times S\varrho + \gamma \times S\sigma + \&c. = S\tau$ nearly.

3. In the Meditationes are aſſumed for π, ϱ, σ, &c. the quantities p, $2p$, $3p$, $4p$, $\ldots \overline{n-2}p$, $\overline{n-1}p$, and np for τ; which, if ſubſtituted for their values in the preceding equations, will give $\alpha + 2\beta + 3\gamma + 4\delta + \&c. = n$, $\alpha + 4\beta + 9\gamma + 16\delta + \&c. = n^2$, $\alpha + 8\beta + 27\gamma + \&c. = n^3$, $\alpha + 16\beta + 81\gamma + \&c. = n^4$; and if the ſums of the ſeries $ax + bx^2 + cx^3 + \&c.$ which reſpectively correſpond to the values p, $2p$, $3p$, $\ldots \overline{n-1}p$ of x be $S1$, $S2$, $S3$, $S4$, $\ldots S\overline{n-1}$, and the ſum of the ſeries $ax + bx^2 + cx^3 + \&c.$ which correſponds to n value of x be Sn; then will

$$Sn = nS\overline{n-1} - n.\frac{n-1}{2}S\overline{n-2} + n.\frac{n-1}{2}.\frac{n-2}{2}S\overline{n-3} \ldots \pm nS1$$

nearly, which equation is given in the above-mentioned Book.

3. The logarithm from the number, the arc from the ſine, &c. are found by ſerieſes of the formula $ax + bx^2 + cx^3 + \&c.$; and conſequently this equation is applicable to them.

4. In the ſame Book is aſſumed a ſeries $ax^r + bx^{r+s} + cx^{r+2s} + dx^{r+3s} + \&c.$ of a more general formula than the preceding, and in it for x ſubſtituted α, β, γ, δ, &c., m; and $S\alpha$, $S\beta$, $S\gamma$, $S\delta$, &c; Sm for the reſulting ſums, and thence deduced

$$Sm = \frac{m^r \times \overline{m^s-\beta^s}\,.\,\overline{m^s-\gamma^s}\,.\,\overline{m^s-\delta^s}\,.\,\&c.}{\alpha^r \times \overline{\alpha^s-\beta^s}\,.\,\overline{\alpha^s-\gamma^s}\,.\,\overline{\alpha^s-\delta^s}\,.\,\&c.} \times S\alpha + \frac{m^r \times \overline{m^s-\alpha^s}\,.\,\overline{m^s-\gamma^s}\,.\,\overline{m^s-\delta^s}\,.\,\&c.}{\beta^r \times \overline{\beta^s-\alpha^s}\,.\,\overline{\beta^s-\gamma^s}\,.\,\overline{\beta^s-\delta^s}\,.\,\&c.}$$

$$\times S\beta + \frac{m^r \times \overline{m^s-\alpha^s}\,.\,\overline{m^s-\beta^s}\,.\,\overline{m^s-\delta^s}\,.\,\&c.}{\gamma^r \times \overline{\gamma^s-\alpha^s}\,.\,\overline{\gamma^s-\beta^s}\,.\,\overline{\gamma^s-\delta^s}\,.\,\&c.} \times S\gamma + \frac{m^r \times \overline{m^s-\alpha^s}\,.\,\overline{m^s-\beta^s}\,.\,\overline{m^s-\gamma^s} \times \&c.}{\delta^r \times \overline{\delta^s-\alpha^s}\,.\,\overline{\delta^s-\beta^s}\,.\,\overline{\delta^s-\gamma^s}\,.\,\&c.}$$

$\times S\delta + \&c.$ nearly.

Cor. If for r and s be aſſumed reſpectively 1, the ſeries becomes $ax + bx^2 + cx^3 + \&c.$ of the ſame formula as the preceding: if $r = 0$ and $s = 1$, the ſeries becomes $a + bx + cx^2 + \&c.$ The latter caſe will be the ſame as the former, when one of the quantities (α) ſubſtituted for x and its correſpondent ſum $S\alpha$, both become $= 0$, and the equation deduced in both caſes the ſame.

5. If π, ρ, σ, &c. reſpectively denote r, $r+p$, $r+2p$, ... $r+\overline{n-2p}$, $r+\overline{n-1p}$, and $\tau = r+np$; and S, $S1$, $S2$, $S3$, ... $S\overline{n-2}$, $S\overline{n-1}$, be the ſums either reſulting from the ſeries $ax + bx^2 + cx^3 + \&c.$ or the ſeries $A + ax + bx^2 + cx^3 + \&c.$, which reſpectively correſpond to the values r, $r+p$, $r+2p$, &c. of x; and Sn the ſum of the ſame ſeries which correſponds to the value $r+np$ of x; then will $Sn = nS\overline{n-1} - n\,.\,\frac{n-1}{2}S\overline{n-2} + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}S\overline{n-3} - \ldots \pm n\,.\,\frac{n-1}{2}S2 \mp nS1 \pm S$ nearly; this equation differs from the preceding by the laſt term S not vaniſhing; in the preceding caſe S became $= 0$, for it was the ſum of the ſeries $ax + bx^2 + cx^3 + \&c.$, which correſponded to $x = 0$.

6. From

6. From the Meditationes it appears, that $r^m - n \times \overline{r \pm p}^m + n \cdot \frac{n-1}{2} \overline{r \pm 2p}^m - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \overline{r \pm 3p}^m +$ &c. to the end of the ſeries $= 0$, if m is leſs than n, and m and n are whole numbers; but if $m = n$, then it will $= \pm 1 \cdot 2 \cdot 3 \cdot 4 \ldots \overline{n-1} \cdot np^m$; whence it is manifeſt, that for the n firſt terms of the ſeries $A + ax + bx^2 +$ &c. the equations are true; and for the $n-1$ firſt terms of the ſeries $ax + bx^2 + cx^3 +$ &c. and in the ſucceſſive term of both the ſerieſes they will err by a quantity nearly $= \pm 1 \cdot 2 \cdot 3 \ldots n \times p^n \times r^{-n} \times$ co-efficient of the term; and the errors of every ſubſequent term (x^{4+n}) will be nearly as $\pm m \cdot \overline{\frac{m-1}{2}} \cdot \overline{\frac{m-2}{3}} \cdot \overline{\frac{m-3}{4}} \ldots \overline{\frac{m-h+1}{h}} \times p^n \times r^{-n} \times$ co-efficient of the term x^{h+n}, if for r, $r+p$, $r+2p$, &c. be ſubſtituted 1, $1 + \frac{p}{r}$, $1 + \frac{2p}{r}$, &c.

7. Let the preceding equation $Sn = nS\overline{n-1} - n \cdot \frac{n-1}{2} S\overline{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot S\overline{n-3} -$ &c. $= n \times \log. \overline{r-p} - n \cdot \frac{n-1}{2} \log. \overline{r-2p} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \log. \overline{r-3p} +$ &c. $= \log. \frac{r \times \overline{r-2p}^s \times \overline{r-4p}^{s'} \times \overline{r-6p}^{s''} \times \&c.}{\overline{r-p}^t - \overline{r-3p}^{t'} \times \overline{r-5p}^{t''} \times \&c.} = \log. K$, where s, s', s'', &c. denote the co-efficients of the alternate terms of the binomial theorem, *viz.* $s = n \cdot \frac{n-1}{2}$, $s' = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$, &c., and $t = n$, $t' = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$, &c. the co-efficients of the remaining alternate terms; the numerator $r \times \overline{r-2p}^s \times \overline{r-4p}^{s'} \times \overline{r-6p}^{s''} \times$ &c. $=$ (if $N = 2^{n-1}$) $r^N - Ppr^{N-1} + Qp^2r^{N-2} - Rp^3r^{N-3} \ldots Lp^{n-1} \times r^{N-n+1} \pm Mp^nr^{N-n} \mp$ &c.; and the denominator $\overline{r-p}^t \times \overline{r-3p}^{t'} \times \overline{r-5p}^{t''} \times$ &c. $= r^N - Ppr^{N-1} + Qp^2r^{N-2} -$

 Rp

$Rp^3r^{n-3} + \ldots Lp^{n-1}r^{N-n+1}(\pm M + 1 \cdot 2 \cdot 3 \ldots \overline{n-1})\, p^n r^{N-n} \mp$ &c., whence the numerator and denominator have the n first terms the same, and the next succeeding terms differ by $1 \cdot 2 \cdot 3 \ldots \overline{n-1}\, p^n r^{N-n}$; the numerator divided by the denominator $= 1 \pm \frac{1 \cdot 2 \cdot 3 \ldots \overline{n-1}}{r^n}\, p^n$ nearly, if r be a great number in proportion to p, &c. it would be $+$ when n is an odd number, and $-$ when even.

8. The logarithm of the fraction K by the common series $= K - 1 - \frac{\overline{K-1}^2}{2} + \frac{\overline{K-1}^3}{3} -$ &c. has for its first term $= \pm \frac{1 \cdot 2 \cdot 3 \ldots \overline{n-1}}{r^n} \times p^n$ nearly; for its second term the square of the first divided by 2, &c.

9. The error of this equation not only depends on the logarithm of K, which may be calculated to any degree of exactness, but in the calculus on the errors of the given logarithms.

10. If r be increased or diminished by any given number, the n first terms of the numerator and denominator will still result the same, and the next succeeding terms will differ by $1 \cdot 2 \cdot 3 \cdot 4 \ldots n-1 \times p^n \times r^{N-n}$.

11. Let $n \cdot \frac{n-1}{2}$ numbers be 2, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$ numbers be 4, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot \frac{n-5}{6}$ numbers be 6, &c.; their sum, the sum of the products of every two, the contents of every three, four, five, &c. to $n-1$ of them will be equal to the sum, the sum of the products of every two, of the contents of every three, four, five, &c. to $\overline{n-1}$ of the following numbers, *viz.* n numbers which are 1, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$ numbers

numbers which are 3, $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5}$, which are 5, &c.; and the sum of the contents of every n of the former will be less than the sum of the contents of every n latter numbers by $1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \overline{n-1}$.

12. The method given in Art. 4. which I name a method of correspondent values, easily deduces and demonstrates the preceding equations, which cannot, without much difficulty, be done by the preceding method of differences; the method of correspondent values is much preferable to the method of differences, both for the facility of its deduction, and the generality of its resolution: for instance, from this method very easily can be deduced, &c. the subsequent and other similar equations.

Ex. 1. $Sn = nS\overline{n-1} - n \cdot \frac{n-1}{2} S\overline{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} S\overline{n-3} \ldots \pm nS1 \mp S$ nearly.

Ex. 2. $S\overline{n+m} = \frac{\overline{m+n} \cdot \overline{m+n-1} \cdot \overline{m+n-2} \ldots \overline{m+2}}{1 \cdot 3 \cdot \cdot n-1} \times S\overline{n-1} - \frac{n-1}{1} \times A \times \frac{m+1}{m+2} \times S\overline{n-2} + \frac{n-2}{2} \times B \times \frac{m+2}{m+3} S\overline{n-3} - \frac{n-3}{3} \times C \times \frac{m+3}{m+4} \times S\overline{n-4} + \frac{n-4}{4} \times D \times \frac{m+4}{m+5} \times S\overline{n-5} -$ &c. nearly, where the letters A, B, C, D, &c. denote the preceding co-efficients, and the converging series is the same as in the preceding example.

Ex. 3. Let the converging series be of the formula $ax + bx^3$ $cx^5 + dx^7 +$ &c.; then will $Sn = \overline{2n-2}\, S\overline{n-1} - \overline{2n-1} \times \frac{2n-4}{2} S\overline{n-2} + \overline{2n-1} \times \frac{2n-2}{2} \times \frac{2n-6}{3} S\overline{n-3} - \overline{2n-1} \cdot \frac{2n-2}{2} \cdot \frac{2n-3}{2} \times \frac{2n-8}{4} S\overline{n-4} +$ &c. nearly, of which the general term is $\overline{2n-1} \cdot \frac{2n-2}{2} \cdot \frac{2n-3}{3} \cdot \cdot \frac{2n-l+1}{l-1} \times \frac{2n-2l}{l} \times S\overline{n-l}$.

Ex. 4. Let the series be of the formula $A + ax^2 + bx^4 + cx^6 +$ &c.; then will $Sn = \frac{n-1}{n} \times \overline{2n-2}\, S\overline{n-1} - \frac{n-2}{n} \times \overline{2n-1} \,.\, \frac{2n-4}{2} \times S\overline{n-2} + \frac{n-3}{n} \times \overline{2n-1} \,.\, \frac{2n-2}{2} \,.\, \frac{2n-6}{3} S\overline{n-3} - \frac{n-4}{n} \times \overline{2n-1} \,.\, \frac{2n-2}{2} \,.\, \frac{2n-3}{3} \times \frac{2n-8}{4} S\overline{n-4} +$ &c. nearly, of which the general term is $\frac{n-l}{n} \times \overline{2n-1} \,.\, \frac{2n-2}{3} \,.\, \frac{2n-3}{3} \dots \frac{2n-l+1}{l-1} \times \frac{2n-2l}{l} \times S\overline{n-l}$.

Ex. 5. Let the given series be of the formula $ax + bx^2 + cx^3$ + &c., and in it for x be substituted p, $-p$, $2p$, $-2p$, $3p$, $-3p$, ..., np, $-np$ and mp, and for the sums of the resulting serieses be wrote respectively S^1, S^{-1}, S^2, S^{-2}, S^3, S^{-3}, ... S^n, S^{-n}, and Sm; then will $Sm =$

$$\frac{m \,.\, \overline{m^2-1} \,.\, \overline{m^2-4} \,.\, \overline{m^2-9} \,.\, \overline{m^2-16} \dots \overline{m^2-\overline{n-1}^2} \,.\, \overline{m-n}}{n \,.\, \overline{n^2-1} \,.\, \overline{n^2-4} \,.\, \overline{n^2-9} \,.\, \overline{n^2-16} \dots \overline{n^2-\overline{n-1}^2} \times 2n = 1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\,.\, 2n}$$

$\times S^{-n} + A \times \frac{m+n}{m-n} S^{+n} - \frac{2n}{1} \times B \times \frac{m-n}{m+n-1} S^{-n+1} - C \times \frac{m+n-1}{m-n-1} \times S^{n-1} + \frac{2n-1}{2} \times D \times \frac{\overline{m-n-1}}{m+n-2} \times S^{-n+2} + \frac{m+n-2}{m-n-2} \times E\, S^{n-2} - \frac{2n-2}{3} \times F \times \frac{\overline{m-n-2}}{m+n-3} S^{-n+3} - G \times \frac{m+n-3}{m-n-3} \times S^{n-3} +$ &c. nearly, where the letters A, B, C, D, &c. respectively denote the preceding co-efficients. In general, the co-efficients of the terms S^{-n+s} and S^{n-s} will be respectively $M = \frac{2n-s+1}{s} \times L \times \frac{\overline{m-n-s+1}}{m+n-s}$ and $M \times \frac{m+n-s}{m-n-s}$, where the letters L and M respectively denote their preceding co-efficients; the co-efficients are to be taken affirmatively, or negatively, according as s is an even or odd number.

Ex. 6. If for x in the preceding series be substituted p, $-p$, $2p$, $-2p$, $3p$, $-3p$, ... $\overline{n-1}p$, $-\overline{n-1}p$, np respectively, then

Sm

$Sm = \frac{m \cdot \overline{m^2-1} \cdot \overline{m^2-4} \cdot \overline{m^2-9} \cdot \overline{m^2-16} \ldots (m^2-(n-1)^2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \ldots (2n-1)} S^n - A \times \frac{m-n}{m+n-1} \times S^{-n+1} - \frac{2n-1}{1} \times B \times \frac{\overline{m+n-1}}{m-n-1} \times S^{n-1} + C \times \frac{\overline{m-n-1}}{m+n-2} \times S^{-n+2} + \frac{2n-2}{2} \times D \times \frac{\overline{m+n-2}}{m-n-2} \times S^{n-2}$ – &c. nearly, when A, B, C, &c. denote as before the preceding co-efficients. The coefficients of the terms S^{-n+s} and S^{n-s} will be respectively $L \times \frac{m-\overline{n-s+1}}{m+n-s}$ and $\frac{2n-1}{s} \times M \times \frac{\overline{m+n-s}}{m-n-s}$, L and M denoting the preceding co-efficients, which are to be taken negatively or affirmatively, as s is an even or an odd number. In this series when $x=0$, the correspondent sum $=0$.

Ex. 7. Let the given series be of the formula $a+bx+cx^2+dx^3+$ &c.; and in it for x be substituted o, p, $-p$, $2p$, $-2p$, $3p$, $-3p \ldots np$, $-pp$ and mp, and for the sums of the resulting serieses be wrote as before S^o, S^1, S^{-1}, S^2, S^{-2}, ... S^n, S^{-n}, and S^m; then will $Sm = \frac{m \cdot \overline{m^2-1} \cdot \overline{m^2-4} \cdot \overline{m^2-9} \ldots (m^2-\overline{n-1}^2) \times \overline{m-n}}{n \cdot \overline{n^2-1} \cdot \overline{n^2-4} \cdot \overline{n^2-9} \ldots (n^2-\overline{n-1}^2) \times 2n} \times S^{-n} + A \times \frac{m+n}{m-n} \times S^{+n} - \frac{2n}{1} \times B \times \frac{m-n}{m+n-1} S^{-n+1}$ – &c. this series observes the same law as the series given in Ex. 5. and only differs from it by the last term So not vanishing, that is, being $=0$.

Ex. 8 Let the series be of the preceding formula $a+bx+cx^2+dx^3+$ &c., and in it for x be substituted o; p, $-p$; $2p$, $-2p$; $3p$, $-3p$; ... $\overline{n-1}p$, $-\overline{n-1}p$, np, and mp, and the sums resulting be So, $S1$, S^{-1}, S^2, S^{-2}, .. $S^{\overline{n-1}}$, S^{-n+1}, S^n and S^m; then will $Sm = \frac{m \cdot \overline{m^2-1} \cdot \overline{m^2-4} \ldots \overline{m^2-(n-1)^2}}{1 \cdot 2 \cdot 3 \cdot 4 \ldots 2n-1} S^n - A \times \frac{m-n}{m+n-1} \times S^{-n+1}$ – &c. the same series as in Ex. 6. and differs from it only by the last term So not vanishing.

Ex. 9. Let the series be of the same formula $a+bx+cx^2+dx^3+$ &c. and in it for x be substituted p, $-p$, $3p$, $-3p$, $5p$, $-5p$, $7p$, $-7p$, ... np, $-np$ and mp; and the sums resulting be S^1, S^{-1}, S^3, S^{-3}, S^5, S^{-5}, S^7, S^{-7}, ... S^n, S^{-n}, and S^m; then will $S^m =$

$$\frac{\overline{m^2-1}\,.\,\overline{m^2-9}\,.\,\overline{m^2-25}\,.\,\overline{m^2-49}\,\ldots\,\overline{m^2-\overline{n-2}^2}\times m+n}{\overline{n^2-1}\,.\,\overline{n^2-9}\,.\,\overline{n^2-25}\,.\,\overline{n^2-49}\,\ldots\,\overline{n^2-\overline{n-2}^2}\times 2n = 2^n\times 1\,.\,2\,.\,3\,.\,4\,.\,5\,..\,n}$$

$\times S^n - A\times\frac{m-n}{m+n}\times S^{-n} - \frac{n}{1}\times B\times\frac{m+n}{m-n+2}\times S^{n-2} + C\times\frac{m-n+2}{m+n-2}\times S^{-n+2} + \frac{n-1}{2}\times D\times\frac{m+n-2}{m-n+4}\times S^{n-4} - E\times\frac{m-n+4}{m+n-4}S^{-n+4} - \frac{n-2}{3}F\times\frac{m+n-4}{m-n+6}S^{n-6} + \frac{m-n+6}{m+n-6}\times G\times S^{-n+6} + \frac{n-3}{4}\times H\times\frac{m+n-6}{m-n+8}\times S^{n-8} -$ &c. nearly, where the letters A, B, C, D, E, &c. denote the preceding co-efficients of the terms S^n, S^{-n}, S^{n-2}, S^{-n+2}, S^{n-4}, S^{-n+4}, S^{n-6}, &c. respectively. The co-efficients of the terms S^{2n-2s} and S^{-n+2s} will be $M=\frac{n-s+1}{s}\times L\times\frac{m+n-2s+2}{m-n+2s}$ and $N=M\times\frac{m-n+2s}{m+n-2s}$; where L, M, and N denote the co-efficients of the terms immediately preceding each other, that is, of the terms $S^{-n+2s-2}$, S^{n-2s}, and S^{-n+2s}. The sign of the first co-efficient M will be + or −, according as s is even or odd; the second term N will have a contrary sign to the first.

These series may be made to begin from any term, which may be easily found by the method of correspondent values, and the subsequent terms from it by the given law; its preceding terms may be deduced from the same law reversed, that is, by putting the numerators of the fractions multiplied into it for the denominators, and the denominators for the numerators.

From

From theſe different ſerieſes may be formed, by adding two or more terms of the given ſeries together for a term of the required ſeries; which method has been applied to converging ſeries in general in the Meditationes.

13. The method of correſpondent values eaſily affords a reſolution of the problems contained in Mr. BRIGG's or Sir ISAAC NEWTON's method of differences.

Ex. 1. Let the quantity be of the formula $a+bx+cx^2+dx^3$ $+\&c. \ldots x^n=y$, and $n+1$ correſpondent values of x and y be given, *viz.* p, q, r, s, &c. of x; $\mathrm{S}p$, $\mathrm{S}q$, $\mathrm{S}r$, $\mathrm{S}s$, &c. of y; then will $y=\frac{\overline{x-q}\,.\,\overline{x-r}\,.\,\overline{x-s}\,.\,\&c.}{\overline{p-q}\,.\,\overline{p-r}\,.\,\overline{p-s}\,.\,\&c.}\times \mathrm{S}p+\frac{\overline{x-p}\,.\,\overline{x-r}\,.\,\overline{x-s}\,.\,\&c.}{\overline{q-p}\,.\,\overline{q-r}\,.\,\overline{q-s}\,.\,\&c.}\times \mathrm{S}q+$ $\frac{\overline{x-p}\,.\,\overline{x-q}\,.\,\overline{x-s}\,.\,\&c.}{\overline{r-p}\,.\,\overline{r-q}\,.\,\overline{r-s}\,.\,\&c.}\times \mathrm{S}r+\frac{\overline{x-p}\,.\,\overline{x-q}\,.\,\overline{x-r}\,.\,\&c.}{\overline{s-p}\,.\,\overline{s-q}\,.\,\overline{s-r}\,.\,\&c.}\times \mathrm{S}s+\&c.$

The truth of this problem very eaſily appears by writing p, q, r, s, &c. for x in the given ſeries.

All the preceding examples may be applied to this caſe, by writing x for m in the given ſeries; hence the reſolutions of ſeveral caſes of equi-diſtant ordinates by eaſy and not inelegant ſerieſes, amongſt which are included the two caſes commonly given on this ſubject.

14. If a quantity be required, which proceeds according to the dimenſions of x, reduce the above given value of y into a quantity proceeding according to the dimenſions of x, and there reſults $y=\Big(\frac{\mathrm{S}p}{\overline{p-q}\,.\,\overline{p-r}\,.\,\overline{p-s}\,.\,\&c.=\mathrm{A}}+\frac{\mathrm{S}q}{\overline{q-p}\,.\,\overline{q-r}\,.\,\overline{q-s}\,.\,\&c.=\mathrm{B}}$

$$+\frac{\mathrm{S}r}{\overline{r-p}\,.\,\overline{r-q}\,.\,\overline{r-s}\,.\,\&c.=\mathrm{C}}+\frac{\mathrm{S}s}{\overline{s-p}\,.\,\overline{s-q}\,.\,\overline{s-r}\,.\,\&c.=\mathrm{D}}+\&c.\Big)\times x^n-$$

$$\Big(\frac{\mathrm{S}p\times\overline{q+r+s+\&c.}}{\mathrm{A}}+\frac{\mathrm{S}q\times\overline{p+r+s+\&c.}}{\mathrm{B}}+\frac{\mathrm{S}r\times\overline{p+q+s+\&c.}}{\mathrm{C}}+$$

$$\frac{\mathrm{S}s\times\overline{p+q+r+\&c.}}{\mathrm{D}}+\&c.\Big)x^{n-1}+\Big(\frac{\mathrm{S}p\times\overline{qr+qs+rs+\&c.}}{\mathrm{A}}+$$

Sq

$$\frac{Sq \times \overline{pr + ps + rs + \&c.}}{B} + \frac{Sr \times \overline{pq + ps + qs + \&c.}}{C} + \frac{Ss \times \overline{pq + pr + qr + \&c.}}{D} + \&c.)$$

$$\times x^{n-2} - \left(\frac{Sp \times \overline{qrs + \&c.}}{A} + \frac{Sq \times \overline{prs + \&c.}}{B} + \frac{Sr \times \overline{pqs + \&c.}}{C} + \frac{Ss \times \overline{pqr + \&c.}}{D} + \right.$$

$$\left.\&c.\right) x^{n-3} + \&c.$$

The law and continuation of this series is evident to any one versant in these matters from inspection.

These fractions may be reduced to a common denominator by substituting for Sp and A the products $Sp \times P$ and $A \times P$, where $P = \overline{q-r} \cdot \overline{q-s} \cdot \overline{r-s} \cdot \&c.$; for Sq and B the products $Sq \times Q$ and $B \times Q$, where $Q = \overline{p-r} \cdot \overline{p-s} \cdot \overline{r-s} \cdot \&c.$; for Sr and C the products $Sr \times R$ and $C \times R$, where $R = \overline{p-q} \cdot \overline{p-s} \cdot \overline{q-s} \cdot \&c.$; for Ss and D the products $Ss \times S'$ and $C \times S'$, where $S' = \overline{p-q} \cdot \overline{p-r} \cdot \overline{q-r} \cdot \&c.$ &c.

The fractions, in particular cases, will often be reducible to lower terms.

15. Let $y = ax^b + bx^{b+l} + cx^{b+2l} + \&c.$, and the correspondent values of x and y be given as before, then will $y =$

$$\frac{x^b \times \overline{x^l - q^l} \times \overline{x^l - r^l} \times \overline{x^l - s^l} \times \&c.}{p^b \times \overline{p^l - q^l} \times \overline{p^l - r^l} \times \overline{p^l - s^l} \times \&c.} \times Sp + \frac{x^b \times \overline{x^l - p^l} \times \overline{x^l - r^l} \times \overline{x^l - s^l} \times \&c.}{q^b \times \overline{q^l - p^l} \times \overline{q^l - r^l} \times \overline{q^l - s^l} \times \&c.} \times Sq$$

$$+ \frac{x^b \times \overline{x^l - p^l} \times \overline{x^l - q^l} \times \overline{x^l - s^l} \times \&c.}{r^b \times \overline{r^l - p^l} \times \overline{r^l - q^l} \times \overline{r^l - s^l} \times \&c.} \times Sr + \frac{x^b \times \overline{x^l - p^l} \times \overline{x^l - q^l} \times \overline{x^l - r^l} \times \&c.}{s^b \times \overline{s^l - p^l} \times \overline{s^l - q^l} \times \overline{s^l - r^l} + \&c.} \times$$

$$Ss + \&c.$$

This series may in the same manner as the preceding be reduced to terms, proceeding according to the dimensions of x; and the serieses given in the examples may *(mutatis mutandis)* be predicated of it.

16. A more general method of correspondent values is given in the Meditationes, as also the subsequent $y = \frac{\overline{x-q} \cdot \overline{x-r} \cdot \overline{x-s} \cdot \&c.}{\overline{p-q} \cdot \overline{p-r} \cdot \overline{p-s} \cdot \&c.}$

x

$\times Sp + \frac{\overline{x-p} \cdot \overline{x-r} \cdot \overline{x-s} \cdot \&c.}{\overline{q-p} \cdot \overline{q-r} \cdot \overline{q-s} \cdot \&c.} \times Sq + \frac{\overline{x-p} \cdot \overline{x-q} \cdot \overline{x-s} \cdot \&c.}{\overline{r-p} \cdot \overline{r-q} \cdot \overline{r-s} \cdot \&c.} \times Sr +$ &c. as in Ex. 1. $= Sp + (x-p)\left(\frac{1}{p-q} \times Sp + \frac{1}{q-p} \times Sq\right) + (x-p)(x-q)\left(\frac{1}{p-q} \times \frac{1}{p-r} \times Sp + \frac{1}{q-p} \times \frac{1}{q-r} \times Sq + \frac{1}{r-p} \times \frac{1}{r-q} \times Sr\right) + (x-p)(x-q)(x-r)\left(\frac{1}{p-q} \cdot \frac{1}{p-r} \cdot \frac{1}{p-s} \cdot \times Sp + \frac{1}{q-p} \cdot \frac{1}{q-r} \cdot \frac{1}{q-s} \times Sq + \frac{1}{r-p} \cdot \frac{1}{r-q} \cdot \frac{1}{r-s} \times Sr + \frac{1}{s-p} \cdot \frac{1}{s-q} \cdot \frac{1}{s-r} \times Ss\right) - \&c.$

The equality of these two different quantities will easily appear by finding the co-efficients of both, which are multiplied into the same given value of y as Sp, Sq, Sr, &c. and the same power of x; for with very little difficulty they will in general be found equal.

It is evident from this resolution that, giving the ordinates and their respective distances from each other, the value of any other ordinate at a given distance from the preceding, found by this method, will result the same, whatever may be the point assumed from which the abscis is made to begin.

3.

1. Let a series be $Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ of such a formula that if in it for x be substituted $a + b$, there results a series $A \times \overline{a+b} + B \times \overline{a+b}^2 + C \times \overline{a+b}^3 + D \times \overline{a+b}^4 + \&c. = (Aa + Ba^2 + Ca^3 + Da^4 + \&c.) \times (1 + qb + rb^2 + sb^3 + tb^4 + \&c.) + (1 + qa + ra^2 + sa^3 + ta^4 + \&c.) \times (Ab + Bb^2 + Cb^3 + Db^4 + \&c.)$ then will the series $Ax + Bx^2 + Cx^3 + Dx^4 + \&c. = Ax + \frac{2B}{1 \cdot 2}x^2 + \frac{2 \cdot 3C}{1 \cdot 2 \cdot 3}x^3 + \frac{24ABC - 8B^3}{1 \cdot 2 \cdot 3 \cdot 4A^2}x^4 + \frac{36C^2A^2 + 24ACB^2 - 16B^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5A^3}x^5 + \frac{9 \cdot 24A^2BC^2 - 4 \times 24AB^3C}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6A^4}x^6 + \frac{216C^3A^3 + 432A^2B^2C^2 - 384ACB^4 + 64B^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7A^5}x^7 + \&c.;$

+&c.; and the series $1+qx+rx^2+sx^3+tx^4+\&c=1+\frac{B}{A}x+$
$\frac{6CA-2B^2}{1.2A^2}x^2+\frac{18CAB-8B^3}{1.2.3A^3}x^3+\frac{36C^2A^2-8B^4}{1.2.3.4A^4}x^4+$
$\frac{180C^2A^2B-120ACB^3+16B^5}{1.2.3.4.5A^5}x^5+\frac{216C^3A^3+216A^2C^2B^2-288ACB^4+64B^6}{1.2.3.4.5.6A^6}x^6$
+&c.

The terms of these two serieses can easily be deduced by the subsequent method. Let $Kx^{n-2}+L^{n-1}+Mx^n$, be successive terms of the series $Ax+Bx^2+Cx^3+\&c.$, and $K'x^{n-2}+L'x^{n-1}$ successive terms of the series $1+qx+rx^2+sx^3+tx^4+\&c.$; then will $M=\frac{2A^2\times B\times K'+6CAK-2B^2K}{n.\overline{n-1}\times A^2}$ and $L'=\frac{n\times A\times M-B\times xL}{A^2}$.

Cor. 1. Let $B=0$, and the two serieses $Ax+Bx^2+Cx^3+Dx^4+$ &c. and $1+qx+rx^2+\&c.$ become respectively $Ax+\frac{2.3}{2.3}Cx^3+$
$\frac{2^2.3^2}{2.3.4.5}\times\frac{C^2}{A}x^5+\frac{2^3.3^3}{2.3.4.5.6.7}\times\frac{C^3}{A^2}\times x^7+\frac{2^4.3^4}{2.3.4.5.6.7.8.9}$
$\times\frac{C^4}{A^3}x^9+\&c.$, and $1+\frac{2.3}{1.2}\times\frac{C}{A}x^2+\frac{2^2.3^2}{1.2.3.4}\times\frac{C^2}{A^2}x^4+$
$\frac{2^3.3^3}{1.2.3.4.5.6}\times\frac{C^3}{A^3}x^6+\&c.$

If in these serieses for A be substituted 1, and for C be substituted $-\frac{1}{2.3}$, there will result the serieses $x-\frac{x^3}{2.3}+$
$\frac{x^5}{2.3.4.5}-\&c.$, and $1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\&c.$ which give the sine and cosine in terms of the arc x.

Cor. 2. Let $C=0$, and the above-mentioned series $Ax+Bx^2+$ &c. becomes $Ax+\frac{2}{1.2}Bx^2 * -\frac{2^3}{1.2.3.4}\times\frac{B^3}{A^2}x^4-\frac{2^4}{1.2.3.4.5}\times$
$\frac{B^4}{A^3}x^5 * +\frac{2^6}{1.2.3.4.5.6.7}\times\frac{B^6}{A^5}\times x^7+\frac{2^7}{1.2.3.4...8}\times\frac{B^7}{A^6}x^8 * -$
$\frac{2^9}{1.2.3.4..10}\times\frac{B^9}{A^8}x^{10}-\frac{2^{10}}{1.2.3..11}\times\frac{B^{10}}{A^9}x^{11}+\&c.$ The law of

this

this series is, first, that every third term vanishes; and, secondly, the signs of every two successive terms change alternately from + to − and − to +; and, lastly, the co-efficient of the term x^n is $\frac{2^{n-1}}{1.2.3..n} \times \frac{B^{n-1}}{A^{n-2}}$; and the series $1+qx+rx^2$ + &c. becomes $1+\frac{B}{A}x-\frac{2B^2}{1.2A^2}x^2-\frac{2^3B^3}{1.2.3A^3}x^3-\frac{2^3B^4}{1.2.3.4A^4}x^4+\frac{2^4B^5}{1.2.3.4.5A^5}x^5+\frac{2^6B^6}{1.2.3.4.5.6A^6}x^6+\frac{2^6\times B^7}{1.2.3..7A^7}x^7-$ &c. In this series the signs of three successive terms alternately change from + to − and − to +; and the co-efficient of the term x^n is $\frac{2^n\times B^n}{1.2.3.nA^n}$ or $\frac{2^{n-1}\times B^n}{1.2.3..nA^n}$ according as n is divisible by 3 or not.

2. Let a series $1+Px+Qx^2+Rx^3+Sx^4+Tx^5+$ &c. be of such a formula, that if in it for x be substituted $a+b$, there results a series $1+P\times\overline{a+b}+Q\times\overline{a+b}^2+R\times\overline{a+b}^3+S\times\overline{a+b}^4$ + &c. $=(1+Pa+Qa^2+Ra^3+Sa^4+\&c.)\times(1+Pb+Qb^2+Rb^3+Sb^4+\&c.)+(Aa+Ba^2+Ca^3+Da^4+\&c.)\times(Ab+Bb^2+Cb^3+Db^4+\&c.)$, then will the series $Ax+Bx^2+Cx^3+Dx^4+$ &c. $=Ax+Bx^2+\left(\frac{2B^2}{3A}-\frac{PB}{3}+A\times\frac{A^2+P^2}{6}\right)x^3+\left(\frac{2B^3-2PAB^2+A^2\times\overline{A^2+P^2}\times B}{6A^2}\right)x^4+$ &c., and the series $1+Px+Qx^2+Rx^3+$ &c. $=1+Px+\frac{A^2+P^2}{2}x^2+\frac{2AB+P\times\overline{A^2+P^2}}{6}x^3+\frac{4B^2+\overline{A^2+P^2}^2}{24}x^4+$ &c. Let $Kx^{n-2}+Lx^{n-1}+Mx^n$ be successive terms of the series $Ax+Bx^2+Cx^3+$ &c., and $K'x^{n-2}+L'x^{n-1}+M'x^n$ successive terms of the series $1+Px+Qx^2+Rx^3+$ &c.; then will $A\times L+P\times L'=n\times M'$ and $B\times K+Q\times K'=n.\frac{n-1}{2}\times M'$ express the law of the serieses.

Cor.

Cor. Let B = 0, then the series $Ax + Bx^2 + Cx^3 + Dx^4 = A \times (x + \frac{P^2 \times A^2}{2 \cdot 3} x^3 + \frac{(P^2 + A^2)^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^5 + \frac{(P^2 + A^2)^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} x^7 + \&c.)$, and the series $1 + Px + Qx^2 + Rx^3 + \&c. = 1 + Px + \frac{P^2 + A^2}{1 \cdot 2} x^2 + P \times \frac{P^2 + A^2}{1 \cdot 2 \cdot 3} x^3 + \frac{(P^2 + A^2)^2}{1 \cdot 2 \cdot 3 \cdot 4} x^4 + P \times \frac{(P^2 + A^2)^2}{1 \cdot 2 \cdot 3 \ldots 5} x^5 + \frac{(P^2 + A^2)^3}{1 \cdot 2 \cdot 3 \ldots 6} x^6 + \&c.$; the co-efficient of the term x^n will be $(P^2 + A^2)^{\frac{n}{2}}$ or $P \times (P^2 + A^2)^{\frac{n-1}{2}}$, according as n is even or odd.

If in the equations before given for x be substituted $a - b$ instead of $a + b$, then in the other quantities for b substitute $-b$.

3. If in Case 2. the difference between the two quantities $(1 + Pa + Qa^2 + \&c.) \times (1 + Pb + Qb^2 + \&c.)$ and $(Aa + Ba^2 + Ca^2 + \&c.) \times (Ab + Bb^2 + Cb^2 + \&c.)$ is assumed $= 1 + P \times \overline{a + b} + Q \times \overline{a + b}^2 + \&c.$, then in the serieses before given for A, B, C, &c. write respectively $\sqrt{-1}A$, $\sqrt{-1}B$, $\sqrt{-1}C$, &c., and there will result the corresponding serieses.

The same principles may be applied to many other cases.

4. Equations of these formulæ may be useful, when the sums of the serieses correspondent to a value (a) of x are given, and the sums of the series correspondent to a value $(a + b)$ of x is required, b having a small ratio to a: for instance, let the given series be $x - \frac{x^3}{2 \cdot 3} + \frac{x^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^7}{2 \cdot 3 \cdot \cdot 7} + \&c.$; the equation found in the first case is $a + b - \frac{(a+b)^3}{2 \cdot 3} + \frac{(a+b)^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c. = (a - \frac{a^3}{2 \cdot 3} + \frac{a^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.) \times (1 - \frac{b^2}{1 \cdot 2} + \frac{b^4}{1 \cdot 2 \cdot 3 \cdot 4} - \&c.) + (1 - \frac{a^2}{1 \cdot 2} + \frac{a^4}{1 \cdot 2 \cdot 3 \cdot 4} - \&c.) \times (b - \frac{b^3}{2 \cdot 3} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.)$;

but

but $a - \frac{a^3}{2 \cdot 3} + \frac{a^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.$, and $1 - \frac{a^2}{1 \cdot 2} + \frac{a^4}{2 \cdot 3 \cdot 4} - \&c.$ are the fine (s) and cofine (c) of an arc a of a circle whofe radius is 1; and, confequently, if the fine s and cofine c of an arc a be given, the fine of an arc $(a+b) = s \times (1 - \frac{b^2}{2} + \frac{b^4}{24} - \&c.) + c(b - \frac{b^3}{2 \cdot 3} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.)$, which feries, if b be very fmall in proportion to a, converges much fafter than the common feries for finding the fine from the arc: it has been given from different principles in the Meditationes, and is alfo eafily deducible from the feries for finding the fine and cofine from the arc by the propofitions ufually given in plane trigonometry: the cofine of the fame arc $(a+b) = c \times (1 - \frac{b^2}{1 \cdot 2} + \frac{b^4}{2 \cdot 3 \cdot 4} - \&c.) - s \times (b - \frac{b^3}{1 \cdot 2 \cdot 3} + \frac{b^5}{1 \cdot 2 \cdot \cdot 5} - \&c.)$,

Ex. 2. Let the feries be $\overline{a+b} + \frac{\overline{a+b}^3}{2 \cdot 3} + \frac{\overline{a+b}^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. = (a + \frac{a^3}{2 \cdot 3} + \frac{a^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. \times (1 + \frac{b^2}{1 \cdot 2} + \frac{b^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.) + (1 + \frac{a^2}{1 \cdot 2} + \frac{a^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.) \times (b + \frac{b^3}{1 \cdot 2 \cdot 3} + \frac{b^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c.)$; but $a + \frac{a^3}{1 \cdot 2 \cdot 3} + \&c.) = x$, and $1 + \frac{a^2}{1 \cdot 2} + \frac{a^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c. = \sqrt{1+x^2}$, if a be the hyperbolic log. of $x + \sqrt{1+x^2}$; therefore $a + b + \frac{\overline{a+b}^3}{2 \cdot 3} + \frac{\overline{a+b}^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. = x \times (1 + \frac{b^2}{2} + \frac{b^4}{2 \cdot 3 \cdot 4} + \&c.) + \sqrt{1+x^2} \times (b + \frac{b^3}{2 \cdot 3} + \&c.$

Let $b + \frac{b^3}{2 \cdot 3} + \frac{b^4}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. = y$, and $(x + \sqrt{1+x^2} \times (y + \sqrt{1+y^2}) = V$, then will $\overline{a+b} + \frac{\overline{a+b}^3}{2 \cdot 3} + \frac{\overline{a+b}^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c. = \overline{\tfrac{1}{2} V - \frac{1}{V}}$.

 5. Let

5. Let a quantity P be a function of x, or the fluent of a function of $x \times \dot{x}$, and the value X of it when $x=a$ be known, and the value of it when $x=a+b$ be required. Find a ſeries of which the firſt term is X, and which proceeds according to the dimenſions of b, if b be a very ſmall quantity, and in general at leaſt ſo ſmall that the ſeries from $x=a$ to $x=a+b$ neither becomes infinite or 0.

In the ſame manner, if an algebraical or fluxional equation or equations, expreſſing the relations between x, y, z, v, &c. be given, find the correſpondent values of y, z, v, &c. to $x=a$, which let be Y, Z, V, &c.; then find ſerieſes for y, z, v, &c. of which the firſt terms let be Y, Z, V, &c. reſpectively, and which proceed according to the dimenſions of b, but ſubject to the ſame conditions as in the preceding caſe.

From fluxional equations may be deduced ſeries which expreſs the value of y, &c. in terms of x, and always diverge, or always converge, whatever may be its value, as appears from the Meditationes.